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On a Covariant Determination of Mass Scales in Warped Backgrounds

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Abstract

We propose a method of determining masses in brane scenarios which is independent of coordinate transformations. We apply our method to the scenario of Randall and Sundrum (RS) with two branes, which provides a solution to the hierarchy problem. The core of our proposal is the use of covariant equations and expressing all coordinate quantities in terms of invariant distances. In the RS model we find that massive brane fields propagate proper distances inversely proportional to masses that are not exponentially suppressed. The hierarchy between the gravitational and weak interactions is nevertheless preserved on the visible brane due to suppression of gravitational interactions on that brane. The towers of Kaluza-Klein states for bulk fields are observed to have different spacings on different branes when all masses are measured in units of the fundamental scale. Ratios of masses on each brane are the same in our covariant and the standard interpretations. Since masses of brane fields are not exponentiated, the fundamental scale of higher-dimensional gravity must be of the order of the weak scale.

I. INTRODUCTION

Brane scenarios with nontrivial gravity backgrounds have recently attracted a lot of attention. We will focus on the proposal of Randall and Sundrum [1] with a single extra dimension and two branes, but our arguments can be applied to any model with warped geometry. The standard way of calculating masses of brane and bulk fields, which we will

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review shortly, is to absorb metric factors by a field redefinition such that the kinetic terms are canonically normalized.

We consider observers that live on branes and assume that the brane metric is the induced metric

$$g_{\rm brane} = G|_{y=y_{\rm brane}}, \tag{1.1}$$

where G is the metric on the full extra dimensional space. The case of observers that live in the bulk will be discussed in a forthcoming paper. We will use x^μ to denote four dimensional coordinates, and y the fifth dimension. Brane observers, by definition, measure distances along the brane using the induced metric. The two mass parameters one needs to determine in any brane scenario are the masses of brane fields that do not propagate in extra dimensions and the scale of gravitational interactions.

We propose to determine all masses by using covariant equations and then expressing all coordinate distances in terms of the proper distances. To determine the masses of brane fields we use the two-point function and measure the rate of exponential falloff in the Euclidean domain. For gravitational interactions, we use linearized perturbations around the background to determine the acceleration of test particles infalling into point-like sources.

Using this covariant procedure we find that brane masses are independent of the warp factor. That is brane fields with Lagrangian mass M are measured by brane observers to have mass M no matter where the brane is located. Newton's constant observed on different branes is proportional to the warp factor on the given brane. This follows the intuitive picture of suppression of gravitational interaction due to the graviton wavefunction, which coincides with the warp factor. Thus, the hierarchy on the visible brane is realized the same way it is realized in the conventional interpretation. That the fundamental scale of higher dimensional gravity is exponentially smaller than that of four dimensional gravity was presented as one of two alternative interpretations in [1]. The covariant interpretation indicates that this is the proper interpretation, and follows if one demands that the same units be used to measure quantities on both branes and in the bulk.

The covariant interpretation allows for straightforward analysis of relative scales on the branes. Consider the following toy model. In accord with the proposed solution to the hierarchy problem include actions on the visible and hidden branes with dimensionful parameters given in terms of one mass scale only, M_5, the scale of the underlying five-dimensional gravity. For simplicity both actions are identical and contain a scalar field of mass parameter M_5 and a gluon field with the same dimensionless coupling $g(M_5)$. There is also a bulk scalar field of mass M_5. What is the spectrum of this theory? The covariant approach gives the answer immediately. Each brane-scalar field gives a spinless particle of mass M_5 in its own brane. The gluons give glueballs, with the same spectrum on both branes, at a mass scale $\Lambda_{\rm QCD} = M_5 \exp\left(-\int_\infty^{g(M_5)} dg/\beta(g)\right)$. Gravity and the bulk-scalar field finally give a difference between the two branes. The effective strength of gravity in the hidden brane is the same as in the underlying five dimensional gravity, but in the visible brane gravity is exponentially weaker. Similarly, to an observer on the hidden brane there is a tower of states with exponentially suppressed masses while an observer on the visible brane sees these excitations as unsuppressed. The same units have been used to describe the spectrum on both visible and hidden branes which is important since meaningful comparisons of masses between the two branes are possible through gedanken experiments.

Before we present any further details let us briefly review the RS model. Consider a Z_2 5-dimensional orbifold with 3-branes at the fixed points. Furthermore, assume there is a negative cosmological constant Λ and tension on the branes, $V_{\rm hid}$ and $V_{\rm vis}$. Then the metric

$$ds^2 = G_{MN}dx^M dx^N = a^2(y)\eta_{\mu\nu}dx^\mu dx^\nu - dy^2 \tag{1.2}$$

solves Einstein's equations provided

$$V_{hid} = -V_{vis} = 24M^3k \qquad \Lambda = -24M^3k^2 \qquad a(y) = e^{-k|y|} \tag{1.3}$$

where M is the fundamental 5-dimensional gravitational mass scale. The brane with negative tension, V_{vis}, contains the visible universe and is located at $y = y_c$ while the hidden brane, with tension V_{hid}, is at $y = 0$. Now consider a scalar field φ on the visible brane. Its action is of the form

$$S_{\rm vis} = \int d^4x\, \sqrt{-g}(\tfrac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \varphi^4 V(\varphi/M)). \tag{1.4}$$

The only available mass scale is M, and it is assumed that the function $V(z)$ does not contain any anomalously large or small numerical constants. Using the four dimensional part of the metric Eq. (1.2) the action is

$$S_{\rm vis} = \int d^4x\, a^4(y_c)(\tfrac{1}{2}a^{-2}(y_c)\eta^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \varphi^4 V(\varphi/M)). \tag{1.5}$$

The kinetic energy is not canonically normalized. Rescaling the fields $\varphi \to \varphi/a(y_c)$ one obtains

$$S_{\rm vis} = \int d^4x\, (\tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \varphi^4 V(\varphi/a(y_c)M)). \tag{1.6}$$

The only scale that appears in this action is $a(y_c)M$, which is exponentially small compared to M provided $ky_c > 1$.

To complete the argument that a hierarchy has been generated Randall and Sundrum verify that the effective 4-dimensional Planck mass $M_{\rm Pl}$ is not exponentially small compared to M. To this effect the four dimensional zero mode of the metric $\bar{h}_{\mu\nu}$ is introduced through

$$ds^2 = a^2(y)(\eta_{\mu\nu} + \bar{h}_{\mu\nu})dx^\mu dx^\nu - dy^2. \tag{1.7}$$

The four dimensional metric is $\bar{g}_{\mu\nu} = \eta_{\mu\nu} + \bar{h}_{\mu\nu}$. The curvature term in the 5-dimensional action contains the four dimensional curvature term, so the effective four dimensional theory, at distances large compared to the size of the fifth dimension is

$$S_{\rm eff} = \int d^4x \int_0^{y_c} dy\; 2M^3a^2(y)\sqrt{-\bar{g}}\bar{R} \tag{1.8}$$

Since $\bar{g}$ is y independent the integral over y is trivial and gives

$$M_{\rm Pl}^2 = \frac{M^3}{k}[1 - a^2(y_c)]. \tag{1.9}$$

Since k is naturally of order M, then also $M_{\rm Pl} \sim M$.

It is apparent that the above interpretation of masses for brane fields depends on the choice of coordinates. For example, one can transform coordinates globally, not just on the brane, such that the masses of brane fields are unchanged on both branes. Take

$$x'^{\mu} = f(y) x^{\mu} \qquad y' = y, \tag{1.10}$$

and choose the function $f(y)$ to satisfy

$$f(y) = \begin{cases} 1 & \text{if } y < y_c/2 - \epsilon \\ a(y_c) & \text{if } y > y_c/2 + \epsilon \end{cases} \tag{1.11}$$

and to interpolate smoothly between these values for $|y - y_c/2| \leq \epsilon$. Then, in the new coordinate system the actions $S_{\rm vis}$ as well as $S_{\rm hid}$ are canonically normalized. Therefore no mass rescaling takes place on either brane. Notice that the transformation is consistent with the orbifold construction (it leaves the branes fixed).

In the new coordinates the 5-dimensional metric fails to be diagonal only in the region $|y - y_c/2| < \epsilon$. As long as we are not interested in gravity, the brane observers would not be aware that the metric is quite complicated in the bulk.

The plan of the paper is as follows. First, in Sec. II we will explain why the masses in brane actions are not exponentially suppressed in *any* coordinate system. Next, in Sec. III we show that Newton's constant depends on the position of the brane. Roughly speaking, Newton's constant is proportional to the warp factor at the position of the brane. In Sec. IV we consider bulk scalar fields. We show that the towers of Kaluza-Klein states are measured to have different masses on the two branes. It may be surprising that the same bulk state is observed by different observers to have different mass, but this is nothing else but the gravitational red shift. For completeness, in the Appendix we derive the Green function of massless and massive bulk scalar fields.

None of the calculations in the paper are original, with the possible exception of the simple derivation in the Appendix. In Sec. III we rely on the results of Refs. [2,3]. The derivation in the Appendix is a simple modification of a result from Ref. [6] to the case with two branes.

II. RESCALING AND MASSES ON THE BRANES

Consider a flat four dimensional space with metric

$$ds^2 = \eta_{\mu\nu} dx^{\mu} dx^{\nu}. \tag{2.1}$$

We would like to study particle propagation in this theory, and to compare results computed with this metric and those obtained in a different coordinate system $x^{\hat{\mu}} = \delta^{\hat{\mu}}_{\mu} \lambda x^{\mu}$, so that

$$ds^2 = \lambda^2 \eta_{\hat{\mu}\hat{\nu}} dx^{\hat{\mu}} dx^{\hat{\nu}}. \tag{2.2}$$

A free scalar field ϕ has action

$$S = \int d^4x \; (\tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}m^2\phi^2) \tag{2.3}$$

$$= \int d^4\hat{x} \; \lambda^4(\tfrac{1}{2}\lambda^{-2}\eta^{\hat{\mu}\hat{\nu}}\partial_{\hat{\mu}}\phi\partial_{\hat{\nu}}\phi - \tfrac{1}{2}m^2\phi^2) \tag{2.4}$$

Rescaling the field, $\phi \to \lambda^{-1}\phi$ the kinetic energy is canonically normalized,

$$S = \int d^4\hat{x} \; (\tfrac{1}{2}\eta^{\hat{\mu}\hat{\nu}}\partial_{\hat{\mu}}\phi\partial_{\hat{\nu}}\phi - \tfrac{1}{2}(\lambda m)^2\phi^2). \tag{2.5}$$

It would seem the field describes a particle of mass λm. However, consider the two-point function obtained by solving the covariant equation

$$(\Box + m^2)G(x,x') = \frac{\delta^4(x-x')}{\sqrt{g}}, \tag{2.6}$$

where $\Box$ is the scalar Laplacian in the background of (2.2). The corresponding Feynman propagator $\Delta^{(4)}(\hat{x},\hat{x}')$ is a function of $\hat{\sigma} \equiv \eta_{\hat{\mu}\hat{\nu}}(x-x')^{\hat{\mu}}(x-x')^{\hat{\nu}}$ only,

$$\Delta^{(4)}(\hat{x},\hat{x}') = \frac{i}{8\pi}\sqrt{\frac{(\lambda m)^2}{-\hat{\sigma}+i\epsilon}}H_1^{(2)}(\sqrt{(\lambda m)^2(\hat{\sigma}-i\epsilon)}), \tag{2.7}$$

where $H_1^{(2)}$ is a Hankel function. We have included the small imaginary part, $i\epsilon$, to remind us that $\Delta^{(4)}$ is really the boundary value of a function that is analytic in the lower $\hat{\sigma}$ complex plane. One can extract the mass by looking for the exponential fall-off in the Euclidean domain, $\Delta^{(4)} \sim \exp(-\lambda m\sqrt{-\hat{\sigma}})$. In terms of the physical Euclidean separation $d_E = \sqrt{-\sigma} = \lambda\sqrt{-\hat{\sigma}}$, the exponential fall-off is $\exp(-md_E)$. The Green function falls-off exponentially over a physical length scale $1/m$, not $1/\lambda m$. As we have stressed already, it is crucial that we do use the background metric to convert to physical, coordinate independent, distances.

The situation is entirely analogous in the RS model. Identifying $\lambda = a(y_c) = \exp(-ky_c)$ the action on the visible brane, Eq. (1.5), has all masses rescaled $M \to \lambda M$ as in the previous section and describes propagation in a background $ds^2 = a^2(y_c)\eta_{\mu\nu}dx^\mu dx^\nu$. One can infer the physical mass by studying propagation and looking for the exponential fall-off in physical separation concluding that the mass is order M rather than $a(y_c)M$, or more simply by rescaling $x \to \exp(ky_c)x$, so

$$ds^2 = e^{2k(y_c-|y|)}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2. \tag{2.8}$$

We note that in more general situations rescaling of the field is not sufficient to bring the action on the visible brane into standard form, while a change of coordinates does. There exist background solutions [4] where the time and space components of the metric are not identical. One can have

$$ds^2 = n^2(y,t)dt^2 - a^2(y,t)\delta_{ij}dx^i dx^j - b^2(y,t)dy^2. \tag{2.9}$$

with $a^2 \neq n^2$ being functions of y and t. For static backgrounds, see for instance Ref. [5], the brane is flat and the effective action for a brane scalar field is

$$S_{\rm vis} = \int d^4x a(y_c)^3 n(y_c) \left[\frac{1}{2}\left(\frac{1}{n^2(y_c)}\partial_t\varphi\partial_t\varphi - \frac{\delta^{ij}}{a^2(y_c)}\partial_i\varphi\partial_j\varphi\right) - \varphi^4 V(\varphi/M)\right]. \qquad (2.10)$$

It is apparent that no field rescaling will bring the action into the canonical form. However, a coordinate transformation can be performed to this effect. More generally, if the 3-brane is flat there is a coordinate system for which the metric is (2.1).

III. PLANCK SCALE FOR BRANE OBSERVERS

We want to find out the effect of point masses placed on branes on test particles also placed on the branes. Assuming that the sources do not significantly affect the background one can perform a computation of linearized perturbations of the metric. For the RS scenario, such a calculation has been presented in Refs. [2,3]. Compare also Ref. [6] for the linearized gravity calculation in the one-brane RS model [7]. Ref. [2] restricts the calculation to metric fluctuations of spin two neglecting the scalar excitations. Ref. [3] shows that the scalar modes effectively do not contribute to long distance forces provided that the radius is stabilized [8,9].

We are only interested in the long distance gravitational interactions, we can therefore neglect all higher Kaluza-Klein states as well as massive fields with spins lower than two. It is however crucial that there are no such massless fields coming from the dimensional reduction of the five-dimensional metric tensor. If there were such massless fields they might provide dominant contribution to the gravitational interactions and actually destroy the hierarchy. For example, the radion coupling on the visible brane are enhanced compared to the spin-two excitations [10] and the radion, if massless, would provide the dominant force at long distance. Of course, if this was the case the model would be ruled out because scalar gravity does not bend light.

Our strategy is as follows. Newton's constant (in four dimensions) G_N is defined, in the non-relativistic limit, as the proportionality constant that gives the acceleration of a particle in the gravitational field of a point mass m,

$$\frac{d^2\vec{x}}{dt^2} = -G_N m \frac{\vec{x}}{|\vec{x}|^3}. \qquad (3.1)$$

We will compute this equation taking the non-relativistic approximation of the geodesic equation for the source and a particle both constrained to one of the branes.

In terms of the metric perturbations

$$ds^2 = (a^2(y)\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu - dy^2 \qquad (3.2)$$

the authors of Refs. [2,3] find that the long distance interactions are governed by

$$\frac{1}{a^2_{h,v}}\Box^{(4)}\bar{h}^{h,v}_{\mu\nu}(y=0,y_c) = -16\pi\, G \sum_{b=h,v} a_b^2 \left[T^b_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^b_{\sigma\rho}\eta^{\sigma\rho}\right], \qquad (3.3)$$

where $b = h, v$ indicates quantities evaluated on the hidden or visible branes, $\Box^{(4)} = \eta^{\mu\nu}\partial_\mu\partial_\nu$, T is the energy-momentum tensor and $G = \left(2M_5^3 \int_{y=0}^{y=y_c} a^2(y)dy\right)^{-1}$. The transverse traceless part of h is denoted by $\bar{h}$.

We are interested in static point sources, for which

$$T^{\mu\nu} = m\, U^\mu U^\nu \delta^{(3)}(x)/\sqrt{-g^{(3)}}. \tag{3.4}$$

Here m is the mass of the source particle, $U^\mu = dx^\mu/d\tau$ is its four velocity, which we take to be only in the time direction. Therefore on $y = 0$ one has $U^0 = a^{-1}(0)$, while on $y = y_c$ it is $U^0 = a^{-1}(y_c)$:

$$T_{h\,00} = m_o \delta^{(3)}(x) \qquad T_{v\,00} = m_c a^{-1}(y_c)\delta^{(3)}(x). \tag{3.5}$$

Putting a source at $\vec{x} = 0$ on either brane we obtain

$$\bar{h}_{00}(x) = m_{h,v} 2G a^3_{h,v} \frac{1}{|\vec{x}|}. \tag{3.6}$$

For a particle motion restricted to the brane we are only interested in the geodesic equations for the x^μ components. Since the metric is static, the only nonzero component of the affine connection linear in the perturbation is $\Gamma^i_{00} = -\frac{1}{2a^2}\bar{h}_{00,i}$. Therefore, the geodesic equation reads

$$\frac{d^2\vec{x}}{d\tau^2} = -m_{h,v} G a_{h,v} \frac{\vec{x}}{|\vec{x}|^3} \left(\frac{dt}{d\tau}\right)^2. \tag{3.7}$$

We need to express the distances in terms of the physical distance $\vec{x}_{\rm phys} = a_{h,v}\vec{x}$. Finally we obtain for the physical acceleration towards the source

$$\frac{d^2\vec{x}_{\rm phys}}{dt^2} = -m_{h,v} G a^2_{h,v} \frac{\vec{x}_{\rm phys}}{|\vec{x}_{\rm phys}|^3}, \tag{3.8}$$

which implies that the effective Newton's constant measured by brane observers are

$$G_N^{h,v} = a^2_{h,v} G. \tag{3.9}$$

On the surface this result follows directly from Eq. (3.3), but the derivation shows that many additional warp factors come in and conspire to give Eq. (3.9) only when describing the acceleration in terms of physical lengths.

Comparing this result with the previous section we see that on the hidden brane all interactions are governed by the same fundamental scale. On the visible brane, brane fields have masses equal to the fundamental scale, while the Newton's constant is suppressed compared to the fundamental scale. Alternatively, the Planck scale inferred by an observer on the visible brane is enhanced compared to the fundamental scale of five-dimensional gravity.

If we want this model to reproduce the observed hierarchy we need to set the fundamental scale of five-dimensional quantum gravity to be of the order of the weak scale. Of course, this situation is similar to the models with large extra dimensions [11]. The crucial distinction between the RS scenario and the large extra dimensions is that the size of the extra dimension in the RS model is only a small multiple of the fundamental scale. Therefore, only a small fine tuning is required to obtain the proper hierarchy (apart from tuning the brane tensions to the bulk cosmological constant). It has already been noted in Ref. [1] that higher-dimensional terms in the Lagrangian are suppressed by powers of the weak scale, so experiments in the near future should see signals of new physics.

IV. KALUZA-KLEIN MODES

For completeness we study the masses of bulk scalar fields. We want to make sure that the hierarchy on the visible brane is not upset by an emergence of a new mass scale [12]. Our strategy is similar to that presented in Section. II. We analyze the full two-point function of a scalar bulk field. We then restrict the full Green function to observations on the branes, that is, with both arguments of the Green function set to either $y = 0$ or $y = y_c$. We then express the x^μ coordinates in terms of invariant distances on the relevant brane.

We therefore turn our attention to the Green function of the Klein-Gordon equation,

$$(\Box + m^2)\Delta(x, y; x', y') = \frac{\delta^4(x - x')\delta(y - y')}{\sqrt{G}}. \tag{4.1}$$

Here $\Box = \frac{1}{\sqrt{G}}\partial_A\sqrt{G}G^{AB}\partial_B$ is the scalar Laplacian in the space (1.2). For simplicity we first consider the case with $m = 0$; $m \neq 0$ is presented before the end of this section. The details of the derivation are relegated to the Appendix. As shown there

$$\Delta(x, y; x', y') = \int \frac{d^4q}{(2\pi)^4} e^{iq\cdot(x-x')}\Delta_q(y, y'), \tag{4.2}$$

where

$$\Delta_q(y, y') = \frac{\pi k^3(zz')^2}{2} \frac{[N_1(\hat{q}z_2)J_2(\hat{q}z_>) - J_1(\hat{q}z_2)N_2(\hat{q}z_>)][N_1(\hat{q}z_1)J_2(\hat{q}z_<) - J_1(\hat{q}z_1)N_2(\hat{q}z_<)]}{N_1(\hat{q}z_1)J_1(\hat{q}z_2) - J_1(\hat{q}z_1)N_1(\hat{q}z_2)}. \tag{4.3}$$

Here J_n and N_n are Bessel functions of the first kind and Neumann functions, respectively, and $\hat{q} \equiv \sqrt{\eta_{\mu\nu}q^\mu q^\nu}$. We have introduced the conformal variable

$$z = \frac{1}{k}e^{ky}, \tag{4.4}$$

the brane values $z_{1,2} = z|_{y=0,y_c}$, and the notation $z_>$ and $z_<$ to represent the larger and smaller of z and z', respectively.

The function $\Delta_q(y, y')$ has isolated poles at $\hat{q} = 0$ and at $\hat{q} = m_n > 0$, where m_n are solutions to

$$N_1(m_n z_1)J_1(m_n z_2) - J_1(m_n z_1)N_1(m_n z_2) = 0. \tag{4.5}$$

It is easy to verify that, for low excitation number n, $m_n \sim a(y_c)k$. Denoting the residues of the poles by $-R_n(y, y')/2m_n$ we have

$$\begin{aligned} \Delta(x, y; x', y') &= -\sum_n \int \frac{d^4q}{(2\pi)^4} \frac{e^{iq\cdot(x-x')}}{q^2 - m_n^2} R_n(y, y') \\ &= \sum_n \Delta^{(4)}(x - x'; m_n) R_n(y, y'), \end{aligned} \tag{4.6}$$

where $\Delta^{(4)}(x - x'; m_n)$ is the four dimensional Green function for a particle of mass m_n. Since the residue factorizes, $R_n(y, y') = r_n(y)r_n(y')$, the full Green function $\Delta(x, y; x', y')$

can be obtained in a four dimensional description by coupling sources $J(x)$ and $J'(x)$ to the linear combinations $\sum_n r_n(y)\psi_n(x)$ and $\sum_n r_n(y')\psi_n(x)$, respectively, where ψ_n is a field of mass m_n.

By the arguments of the previous section, the result of expressing distances in terms of the invariant distances an observer on the visible (negative tension) brane sees particles of physical mass $m_n/a(y_c) \sim k$. Their 'overlap', or wave-function on the visible brane, is given by $r_n(y_c)$. It is only observers on the *hidden* (positive tension) brane who see exponentially suppressed masses. The calculation can be easily repeated using the rescaled metric (2.8), and the same conclusions are reached. There is a simple physical interpretation. Hidden brane observers see masses that have climbed up a potential well and are therefore red-shifted compared to the visible brane precisely by the warp factor.

This discussion goes through with little modification in the case of massive bulk scalars. Using the Fourier transform in (4.2), the solution to Eq. (4.1) is

$$\Delta_q(y,y') = \frac{\pi k^3 (zz')^2}{2} \frac{[\tilde{N}_\nu(\hat{q}z_2) J_\nu(\hat{q}z_>) - \tilde{J}_\nu(\hat{q}z_2) N_\nu(\hat{q}z_>)][\tilde{N}_\nu(\hat{q}z_1) J_\nu(\hat{q}z_<) - \tilde{J}_\nu(\hat{q}z_1) N_\nu(\hat{q}z_<)]}{\tilde{N}_\nu(\hat{q}z_1)\tilde{J}_\nu(\hat{q}z_2) - \tilde{J}_\nu(\hat{q}z_1)\tilde{N}_\nu(\hat{q}z_2)}. \tag{4.7}$$

where z is given by Eq. (4.4), $\nu = \sqrt{4 + m^2/k^2}$, and we have introduced the shorthand

$$\tilde{Z}_\nu(z) = (1 - \frac{\nu}{2}) Z_\nu(z) + \frac{z}{2} Z_{\nu-1}(z), \tag{4.8}$$

for Z a Bessel function. The function $\Delta_q(y,y')$ is regular at $q = 0$. However it diverges at $q = 0$ as $m \to 0$. Therefore, there is no massless particle in the Kaluza-Klein spectrum except for $m = 0$. The spectrum is determined by the zeroes of the denominator,

$$\tilde{N}_\nu(m_n z_1)\tilde{J}_\nu(m_n z_2) - \tilde{J}_\nu(m_n z_1)\tilde{N}_\nu(m_n z_2) = 0. \tag{4.9}$$

This is precisely the same equation as found by Goldberger and Wise by means of a different method [12], namely, direct diagonalization of the action integral. For small m, the low excitation number spectrum has $m_n \sim a(y_c)k$. As above, the full Green function can be written as a sum over poles, Eq. (4.6), where the residues factorize. The physical picture is the same as in the massless case. For m of the order the fundamental scale, the lowest mass state that an observer on the visible brane measures has mass of order m, while an observer on the hidden brane measures a mass of order $a(y_c)m$.

V. SUMMARY

We have discussed a covariant procedure for determining masses in brane scenarios. The results are independent of coordinate rescaling. We analyzed the situation from the point of view of brane observers who measure distances along branes using, as they must, the induced metric. Rather than absorbing the metric into redefinitions of matter fields we soaked up the warp factors by expressing all coordinate quantities in terms of invariant distances.

Applying the covariant approach to the Randall-Sundrum solution of the hierarchy problem one finds that the observed masses for brane fields coincide with their Lagrangian values.

The hierarchy is realized due to a suppression of gravitational interactions on the visible brane. For observers on the visible (negative tension) brane the effective value of Newton's constant is suppressed by the warp factor $a(y)$. This suppression is a result of the overlap of the graviton wave function with the brane fields [1]. It is important for maintaining the hierarchy that the long-range forces are mediated only by the four-dimensional graviton. In the absence of a mechanism for radius stabilization the radion field would provide the dominant contribution to the static gravitational force on the visible brane.

In the covariant approach it is natural to always compare masses with the fundamental scale of the underlying five-dimensional theory M_5. All bulk fields, not only the graviton, look differently to observers on different branes. The towers of Kaluza-Klein states for bulk fields are observed to have different spacings on different branes. Masses measured on different branes are related to each other by the ratio of the corresponding warp factors. This is a result of climbing or falling into the gravitational potential.

The RS model solves the hierarchy problem by naturally assuming that all mass parameters in the underlying Lagrangian, including the brane Lagrangians, must be of order M_5. Since the observed masses for brane fields coincide with their Lagrangian values, M_5 must be around the weak scale to ensure the weak scale vacuum expectation value for the Higgs field.

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APPENDIX A: BULK SCALAR GREEN FUNCTIONS

For completeness we derive the Green function of a massless and a massive bulk scalar field. We then comment on the results of Ref. [12]. We start with the massless case because it is simpler. Our derivation follows the derivation of Ref. [6]. The only difference from Ref. [6] is in the boundary conditions since we are analyzing a two-brane scenario.

In the weak coupling limit the scalar field does not affect the metric. Thus, we solve the equation

$$\left(\frac{1}{\sqrt{G}}\partial_M\sqrt{G}G^{MN}\partial_N\right)\Delta(x,z,x',z') = \frac{\delta^4(x-x')\delta(z-z')}{\sqrt{G}} \tag{A1}$$

with fixed background metric

$$ds^2 = G_{MN}dx^M dx^N = \frac{1}{(kz)^2}(\eta_{\mu\nu}dx^\mu dx^\nu - dz^2). \tag{A2}$$

Here we use conformaly flat coordinates with $z = \frac{1}{k}e^{ky}$. For the calculation in the rescaled metric (2.8) simply take $z = \frac{1}{k}e^{k(y-y_c)}$. With this choice of coordinates the 5-dimensional Laplacian has the form

$$\frac{1}{\sqrt{G}}\partial_M \sqrt{G} G^{MN} \partial_N = (kz)^2 \eta^{\mu\nu} \partial_\mu \partial_\nu - (kz)^5 \partial_z \frac{1}{(kz)^3} \partial_z. \tag{A3}$$

We first Fourier transform the Green function

$$\Delta(x, z, x', z') = \int \frac{d^4q}{(2\pi)^4} e^{iq(x-x')} \Delta_q(z, z'). \tag{A4}$$

In the momentum space Eq. (A1) takes the form

$$\frac{1}{(kz)^3} \left(-q^2 - \partial_z^2 - \frac{3}{z}\partial_z \right) \Delta_q(z, z') = \delta(z - z'). \tag{A5}$$

where $q^2 = \eta^{\mu\nu} q_\mu q_\nu$. Introducing $\hat{\Delta}_q = (\frac{1}{k^2 zz'})^2 \Delta_q$, like in Ref. [6], the above equation becomes the Bessel equation

$$(z^2 \partial_z^2 + z\partial_z + q^2 z^2 - 4)\hat{\Delta}_q = -\frac{z}{k}\delta(z - z'). \tag{A6}$$

The standard method for solving this kind of equation is to first find solutions to the homogeneous equation in two regions $z < z'$ and $z > z'$. Let us call these solutions $\hat{\Delta}_<$ and $\hat{\Delta}_>$, respectively. Then the full equation is solved by requiring that the discontinuity of the first derivative at $z = z'$ reproduces the delta function. The second order Bessel functions $J_2(qz)$ and $N_2(qz)$ are the linearly independent solutions to the homogeneous part of Eq. (A6).

We impose the Neumann boundary conditions at $z = R_<$ and $z = R_>$ on the original Green function $\partial_z \Delta|_{z=R_<,R_>} = 0$. $R_<$ corresponds to the hidden brane, $z = R_>$ to the visible one. We obtain

$$\hat{\Delta}_< = A_<(z') \left[N_1(qR_<) J_2(qz) - J_1(qR_<) N_2(qz) \right], \tag{A7}$$
$$\hat{\Delta}_> = A_>(z') \left[N_1(qR_>) J_2(qz) - J_1(qR_>) N_2(qz) \right]. \tag{A8}$$

We could have imposed the boundary conditions taking the derivatives with respect to z' instead of z and correspondingly exchanged the subscripts $<$ with $>$. Therefore, by symmetry we can write

$$\hat{\Delta}_< = C_< \left[N_1(qR_>) J_2(qz') - J_1(qR_>) N_2(qz') \right] \left[N_1(qR_<) J_2(qz) - J_1(qR_<) N_2(qz) \right], \tag{A9}$$
$$\hat{\Delta}_> = C_> \left[N_1(qR_<) J_2(qz') - J_1(qR_<) N_2(qz') \right] \left[N_1(qR_>) J_2(qz) - J_1(qR_>) N_2(qz) \right], \tag{A10}$$

where $C_<$ and $C_>$ are constants.

The solution has to be continuous at $z = z'$, so

$$C_< = C_> \equiv C. \tag{A11}$$

Moreover, the first derivative must be discontinuous

$$kz\partial_z(\Delta_> - \Delta_<)|_{z=z'} = 1, \tag{A12}$$

which implies that

$$1 = kqzC\Big[\,[N_1(qR_<)J_2(qz) - J_1(qR_<)N_2(qz)]\,[N_1(qR_>)J_2'(qz) - J_1(qR_>)Y_2'(qz)] \tag{A13}$$

$$- [N_1(qR_>)J_2(qz) - J_1(qR_>)N_2(qz)]\,[N_1(qR_<)J_2'(qz) - J_1(qR_<)Y_2'(qz)]\Big]. \tag{A14}$$

Using the identities $zZ'_\nu = zZ_{\nu-1} - \nu Z_\nu$, where Z stands for either J_ν or N_ν and $N_1(z)J_2(z) - J_1(z)N_2(z) = \frac{2}{\pi z}$ we obtain

$$\frac{1}{C} = \frac{2k}{\pi}\left[J_1(qR_>)N_1(qR_<) - N_1(qR_>)J_1(qR_<)\right]. \tag{A15}$$

Solving the massive case is now simple. We need to Fourier transform in the x variables and perform the redefinition from Δ_q to $\hat{\Delta}_q$. The only modification of Eq. (A6) is the mass term

$$(z^2\partial_z^2 + z\partial_z + q^2z^2 - 4 - m^2)\hat{\Delta}_q = -\frac{z}{k}\delta(z-z'). \tag{A16}$$

The solutions to the homogeneous part are $J_\zeta(qz)$ and $N_\zeta(qz)$, where $\zeta = \sqrt{4+m^2}$.

Again, we impose the Neumann boundary conditions at $z = R_<$ and $z = R_>$. When taking derivatives of J_ζ and N_ζ we encounter the linear combinations

$$\tilde{J}_\zeta(x) \equiv (1 - \frac{\zeta}{2})J_\zeta(x) + \frac{x}{2}J_{\zeta-1}(x), \tag{A17}$$

$$\tilde{N}_\zeta(x) \equiv (1 - \frac{\zeta}{2})N_\zeta(x) + \frac{x}{2}N_{\zeta-1}(x). \tag{A18}$$

In terms of these newly defined variables equations for the massive case look similar to the massless ones. Imposing the boundary conditions gives

$$\hat{\Delta}_< = C_<\left[\tilde{N}_\zeta(qR_>)J_\zeta(qz') - \tilde{J}_\zeta(qR_>)N_\zeta(qz')\right]\left[\tilde{N}_\zeta(qR_<)J_\zeta(qz) - \tilde{J}_\zeta(qR_<)N_\zeta(qz)\right], \tag{A19}$$

$$\hat{\Delta}_> = C_>\left[\tilde{N}_\zeta(qR_<)J_\zeta(qz') - \tilde{J}_\zeta(qR_<)N_\zeta(qz')\right]\left[\tilde{N}_\zeta(qR_>)J_\zeta(qz) - \tilde{J}_\zeta(qR_>)N_\zeta(qz)\right]. \tag{A20}$$

In complete analogy to the massless case, continuity at $z = z'$ requires $C \equiv C_< = C_>$. The difference in the first derivatives fixes C to be

$$\frac{1}{C} = \frac{2k}{\pi}\left[\tilde{J}_\zeta(qR_>)\tilde{N}_\zeta(qR_<) - \tilde{N}_\zeta(qR_>)\tilde{J}_\zeta(qR_<)\right]. \tag{A21}$$

As we discussed in Sec. IV the poles of Green functions, or zeros of Eq. (A21), correspond to physical masses after proper rescaling. We want to comment of the results of Ref. [12]. Goldberger and Wise consider a free scalar field bulk action

$$\begin{aligned} S &= \frac{1}{2}\int d^4x dy \sqrt{G}\left(G^{AB}\partial_A\Phi\partial_B\Phi - m^2\Phi^2\right) \\ &= \frac{1}{2}\int d^4x dy \left(e^{-2k|y|}\eta^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + \Phi\partial_y\left(e^{-4k|y|}\partial_y\Phi\right) - m^2e^{-4k|y|}\Phi^2\right), \end{aligned} \tag{A22}$$

The Kaluza-Klein decomposition is in terms of the modes

$$\Phi(x,y) = \sum_n \psi_n(x)\xi_n(y), \tag{A23}$$

which satisfy

$$\int_0^{y_c} dy e^{-2k|y|} \xi_n(y) \xi_m(y) = \delta_{nm} \tag{A24}$$

and

$$-\frac{d}{dy}\left(e^{-4k|y|}\frac{d\xi_n}{dy}\right) + m^2 e^{-4k|y|}\xi_n = m_n^2 e^{-2k|y|}\xi_n. \tag{A25}$$

Inserting this in Eq. (A22) gives

$$S = \frac{1}{2}\sum_n \int d^4x \left[\eta^{\mu\nu}\partial_\mu\psi_n\partial_\nu\psi_n - m_n^2\psi_n^2\right]. \tag{A26}$$

Solving Eq. (A25) for m_n gives precisely our Eq. (A21). The explanation is that any procedure for diagonalizing the action should lead to the same eigenvalues. However, m_n are not coordinate-invariant quantities. In our covariant interpretation, m_n are the physical masses on the hidden brane, while on the visible brane $m_n/a(y_c)$ are the measured masses.

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